JOURNAL OF ALGEBRA 53, 1–20 (1978)

Representations of the Hyperoctahedral Groups

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Communicated by Walter Feit

Received September 27, 1976

INTRODUCTION

We describe here the first part of a systematic development of the representation theory of the hyperoctahedral groups $B(n)$ (Weyl or Coxeter groups of type B (or C)) which emphasises the combinatorial analogies with that of the symmetric groups $S(n)$. If π is a partition of integer k and λ a partition of $n - k$ we call the ordered pair $(\pi; \lambda)$ a double partition of n. As in the case of $S(n)$ and ordinary partitions of n, there are natural bijections between the set of double partitions of n, the set of irreducible representations (denoted $\{\pi; \lambda\}$) of the group $B(n)$, and a set of permutation representations (denoted $\Delta(\pi; \lambda)$) which also form an integral basis for the representation ring. The proof of this latter fact, along with some related computations, lead us to introduce a partial order $\lesssim$ on the set of all $(\pi; \lambda)$ which "extends" the dominance order on ordinary partitions. We show that this order $\lesssim$ has many properties analogous to those known for the dominance order, though it is not a lattice order.

We use (at least implicitly) the results obtained by Young [15] and Osima [12] characterizing the irreducible representations of $B(n)$. We also use frequently well-known facts about irreducible and permutation characters of $S(n)$, especially those contained in Snapper [14] and Liebler and Vitale [10]. In many cases our analogous results for double partitions and characters of $B(n)$, when restricted to degenerate double partitions of the form $(0, \lambda)$, reduce to the original results. Parts of section II, in particular Cor. II. 3, were obtained independently by Mayer [11] and the algorithm stated by him there is clearly a corollary of our Theorem III.5.

In future papers we will derive results for the Weyl group $D(n)$ similar to those of Section II and we will show that inner products involving the $\{\pi; \lambda\}$ and the $\Delta(\pi; \lambda)$ naturally count certain types of integer sequences and also pairs of integral matrices satisfying certain conditions (weights for Z_2 sets). We will also develope an isomorphism between the module of polynomials of total degree n

0021-8693/78/0531-0001$02.00/0

which are symmetric in each of two disjoint sets of variables and the characters of $B(n)$, directly analogous to that defined by Frobenius and Schur for symmetric polynomials and characters of $S(n)$ [16].

I. Characters of the Symmetric Groups

In this section we list several results which we will use frequently from the representation theory of finite groups, especially that of the symmetric group $S(n)$ of degree n. In the following G will be a finite group, H a subgroup of G, and all representations (reps) will be in finite dimensional complex vector spaces. Often "irreducible representation" will be shortened to "irrep." Frequently a representation α and its character χ^α will be identified and so also the representation ring $R(G)$ (Grothendieck ring of representations of G) and the character ring char (G) (integral combinations of irreducible characters). The usual inner (scalar) product of characters $(1/|G|)\sum \chi^\alpha(g)\,\bar{\chi}^\beta(g)$ is denoted by $\langle\chi^\alpha, \chi^\beta\rangle_G$ or $\langle\alpha, \beta\rangle_G$. If σ and η are representations of H and G respectively, $\mathrm{Ind}_H{}^G\,\sigma$ will denote the induced representation of G afforded by σ and $\mathrm{Res}_H{}^G\,\eta$ denotes the restriction of η to a representation of H.

The following basic facts can be found in Curtis and Reiner [1] and Serre [5].

THEOREM I.1 [Frobenius Reciprocity]. *For σ and η representations of H and G, $\langle \mathrm{Ind}_H{}^G\,\sigma, \eta\rangle_G = \langle\sigma, \mathrm{Res}_H{}^G\,\eta\rangle_H$.*

Suppose K and H are subgroups of G and S is a system of representatives for the double cosets HsK in G.

THEOREM I.2 [Mackey Subgroup Theorem]. *For σ a character of H,*

$$\mathrm{Res}_K{}^G\,\mathrm{Ind}_H{}^G\,\sigma = \sum_{s\in S} \mathrm{Ind}^K_{K\cap sHs^{-1}}\,\mathrm{Res}^{sHs^{-1}}_{K\cap sHs^{-1}}\,\sigma^s$$

where $\sigma^s(shs^{-1}) = \sigma(h)$.

From representations σ_i of G_i on V_i one derives the representation $\sigma_1 \otimes \sigma_2$ of $G_1 \times G_2$ of $V_1 \otimes_{\mathbf{C}} V_2$, and it is easy to check that $\langle\sigma_1 \otimes \sigma_2, \eta_1 \otimes \eta_2\rangle_{G_1\times G_2} = \langle\sigma_1, \eta_1\rangle_{G_1} \cdot \langle\sigma_2, \eta_2\rangle_{G_2}$.

THEOREM I.3. *The mapping $(\sigma_1, \sigma_2) \to \sigma_1 \otimes \sigma_2$ is a bijection of the set of all ordered pairs of isomorphism types of irreps of G_1 and G_2 respectively onto the set of all isomorphism types of irreps of $G_1 \times G_2$.*

THEOREM I.4. *If σ_i are representations of subgroups H_i of G_i then $\mathrm{Ind}^{G_1\times G_2}_{H_1\times H_2}\,\sigma_1 \otimes \sigma_2 = \mathrm{Ind}^{G_1}_{H_1}\,\sigma_1 \otimes \mathrm{Ind}^{G_2}_{H_2}\,\sigma_2$.*

We shall say that a sequence $M = (m(1),\dots, m(a))$ of nonnegative integers partitions n, denoted $M \vdash n$, (or say M is an ordered partition of n) if $\sum m(i) = n$. If M is in descending order we will simply call it a partition and usually denote it by a Greek letter. Associate to each such ordered partition M of n the subgroup $S(M)$ of $S(n)$ consisting of all permutations which permute among themselves the elements of the first block of $m(1)$ integers and similarly for the ith block of $m(i)$ integers for all i. If $P \vdash k$ and $Q \vdash n-k$ then $S(P) \times S(Q)$ can naturally be regarded as a subgroup of $S(n)$, namely as the subgroup $S(P \cup Q)$ where $P \cup Q = (p(1),\dots, p(a), q(1),\dots, q(b))$. For partitions π and α, $\pi \cup \alpha$ will denote the above concatenation of ordered sequences as well as the corresponding reordered (descending) partition (denoted by $\pi \otimes \alpha$ by Hall [9]). For partitions α, π of k, r by $\pi \cap \alpha$ we mean the partition gotten by choosing each of the common blocks with multiplicity the minimum of its multiplicities in π and α. That is $\pi \cap \alpha$ is the partition with the largest number of nontrivial parts such that $\pi = (\pi \cap \alpha) \cup \pi'$ and $\alpha = (\pi \cap \alpha) \cup \alpha'$. Following Hall we denote by $P + Q$ the sequence $(p(1) + q(1), p(2) + q(2),\dots)$. The conjugate π^* of a partition π is defined by $\pi^*(j) = |\{k \mid \pi(k) \geqslant j\}|$. Hall notes that $(\pi \cup \alpha)^* = \pi^* + \alpha^*$.

It is well known that to each partition π of n we can associate a unique irrep (character) of $S(n)$ denoted by $\{\pi\}$ (see for example Robinson [4] or Murnaghan [3]). In particular $\{n\}$ is the identity representation and $\{1^n\}$ is the alternating representation (determinant of the canonical n-dim'ℓ rep). Also for any partition π of n, $\{\pi\} \cdot \{1^n\} = \{\pi^*\}$ (Snapper [14, Lemma 9.1]).

Let 1 denote the trivial character for any group. If $M \vdash n$ let $\Delta(M) = \mathrm{Ind}_{S(M)}^{S(n)} 1$, that is, $\Delta(M)$ is the transitive permutation representation afforded by the action of $S(n)$ on the set of left cosets of $S(M)$. We will need the following facts about the irreps $\{\alpha\}$ appearing in $\Delta(\pi)$.

Proposition I.5. *If* $0 \leqslant i \leqslant n - i$ *then*

$$\Delta(n - i, i) = \sum \{n - i + k, i - k\} \qquad (0 \leqslant k \leqslant i).$$

Proof. See Snapper [14, Theorem 4.2, p. 528].

Proposition I.6. $\langle \Delta(\pi), \{n\}\rangle = 1$. *Also* $\langle \Delta(\pi), \{1^n\}\rangle = 1$ *if* π *is* 1^n *and* 0 *otherwise.*

Proof. The first is Burnside's lemma; for the second see Snapper [14, Proposition 6.2, p. 530].

Define a linear order on the partitions of n by letting $\pi \leqslant \alpha$ if $\pi = \alpha$ or there is a k such that $\pi(i) = \alpha(i)$ for $i < k$ and $\pi(k) > \alpha(k)$. This is the dual of the linear order defined in [14].

THEOREM I.7. *For partitions π and α of n,*

$$\langle \Delta(\pi), \{\alpha\} \rangle = 0 \quad \text{if} \quad \pi < \alpha$$

and

$$\langle \Delta(\pi), \{\pi\} \rangle = 1.$$

Proof [14, p. 532].

As noted by Snapper [14], an important consequence of Theorem I.7 is that $\{\Delta(\pi)\colon \pi \vdash n\}$ is an integral basis for the representation ring $R(S(n))$, a classical result of Frobenius.

There is another partial order on the partitions of n which is more closely tied to the permutation characters. Say that α dominates π, denoted by $\alpha \rhd \pi$ or $\pi \lhd \alpha$, if $\sum_1^k \pi(i) \leqslant \sum_1^k \alpha(i)$ for all k [14].

THEOREM I.8. *The following are equivalent:*

(i) $\alpha \rhd \pi$,

(ii) $\langle \Delta(\pi), \{\alpha\} \rangle \neq 0$,

(iii) $\Delta(\pi) - \Delta(\alpha)$ *is* 0 *or a proper character,*

(iv) $\langle \Delta(\pi), \{1^n\} \cdot \Delta(\alpha^*) \rangle \neq 0$.

Proof. See Snapper [14, pp. 531–532] and Liebler and Vitale [10, Theorem 1].

COROLLARY I.9. *If $\alpha \rhd \pi$ then $\alpha \leqslant \pi$.*

The lattice $L(n)$ of partitions of n with the dominance order has been described in much detail by Brylawski [6]. It is frequently useful to know that conjugation of partitions is an antiautomorphism of $L(n)$. (We denote the dual lattice by $L^*(n)$).

COROLLARY I.10. $\alpha \rhd \pi$ *iff* $\pi^* \rhd \alpha^*$.

We also need some new technical results relating the dominance orders in $L(n)$ and $L(m)$ for $n \neq m$.

Let α and π partition n then $\pi \vee \alpha$ (resp $\pi \wedge \alpha$) will denote the supremum (resp. infimum) in the dominance lattice. Brylawski [6] shows that for every j, $\sum_{i=1}^{j} [\pi \wedge \alpha](j) = \min(\sum_{i=1}^{j} \pi(i), \sum_{i=1}^{j} \alpha(i))$.

LEMMA I.11. *For any partition λ*

(i) $(\alpha + \lambda) \wedge (\pi + \lambda) = (\alpha \wedge \pi) + \lambda$,

(ii) $(\alpha \cup \lambda) \vee (\pi \cup \lambda) = (\alpha \vee \pi) \cup \lambda$.

Proof. (i) For any j,

$$\begin{aligned}\sum_{i=1}^{j} [(\alpha+\lambda)\wedge(\pi+\lambda)](i) &= \min\left(\sum_{i=1}^{j}[\alpha+\lambda](i), \sum_{i=1}^{j}[\pi+\lambda](i)\right)\\ &= \min\left(\sum_{i=1}^{j}\alpha(i), \sum_{i=1}^{j}\pi(i)\right)+\sum_{i=1}^{j}\lambda(i)\\ &= \sum_{i=1}^{j}[\alpha\wedge\pi](i)+\sum_{i=1}^{j}\lambda(i)\\ &= \sum_{i=1}^{j}[(\alpha\wedge\pi)+\lambda](i).\end{aligned}$$

For (ii) we have, using (i) and earlier remarks,

$$\begin{aligned}(\alpha\cup\lambda)\vee(\pi\cup\lambda) &= [(\alpha^*+\lambda^*)\wedge(\pi^*+\lambda^*)]^*\\ &= [(\alpha^*\wedge\pi^*)+\lambda^*]^*\\ &= (\alpha\vee\pi)\cup\lambda.\end{aligned}$$

PROPOSITION I.12. *Let $\pi \vdash n$ and $\sigma \vdash n+r-k$ where $r \geqslant k$. If $\sigma\cup k \trianglerighteq \pi \cup r$, then $\sigma \cup (k-j) \trianglerighteq \pi\cup(r-j)$ for $1 \leqslant j \leqslant k$.*

Proof. It is enough to show this for $j=1$.

Choose integers u, v such that $\sigma(u) \geqslant k > \sigma(u+1)$ and $\pi(v) \geqslant r > \pi(v+1)$. Then written in descending order

$$\begin{aligned}\sigma\cup k &= (\sigma(1),\ldots,\sigma(u), k, \sigma(u+1),\ldots),\\ \sigma\cup(k-1) &= (\sigma(1),\ldots,\sigma(u), k-1, \sigma(u+1),\ldots),\\ \pi\cup r &= (\pi(1),\ldots,\pi(v), r, \pi(v+1),\ldots),\\ \pi\cup(r-1) &= (\pi(1),\ldots,\pi(v), r-1, \pi(v+1),\ldots).\end{aligned}$$

It is easy to check that

$$\sum_{1}^{m}(\sigma\cup(k-1))(i) \geqslant \sum_{1}^{m}(\pi\cup(r-1))(i)$$

for $m \leqslant \min(u,v)$ or $m > \max(u,v)$ or $v < m \leqslant u$, and finally with a little more care also the case $u < m \leqslant v$.

PROPOSITION I.13. *If π, α are partitions of n and σ a partition of k, then $\pi \trianglerighteq \alpha$ iff $\pi\cup\sigma \trianglerighteq \alpha\cup\sigma$.*

Proof. It $\pi \trianglerighteq \alpha$ then by Proposition I.8, $\Delta(\alpha)-\Delta(\pi)$ is 0 or a proper character. But by transitivity of induction $\Delta(\alpha\cup\sigma)-\Delta(\pi\cup\sigma) =$

$\mathrm{Ind}_{S(n)\times S(k)}^{S(n+k)}[\Delta(\alpha) - \Delta(\pi)] \otimes \Delta(\sigma)$ since $\Delta(\alpha) \otimes \Delta(\sigma) = \mathrm{Ind}_{S(\alpha)\times S(\sigma)}^{S(n)\times S(k)} 1$ by Proposition I.4. Hence $\Delta(\alpha \cup \sigma) - \Delta(\pi \cup \sigma)$ is 0 or a proper character and so $\pi \cup \sigma \trianglerighteq \alpha \cup \sigma$ by Theorem I.8. (One could of course also give a direct combinatorial proof.)

For the converse it is enough to consider the case of $\sigma = k$, and this is just a special case of Proposition I.12 with $k = r = j$.

COROLLARY I.14. *If β is a partition of b, λ a partition of q, and if $\beta \trianglerighteq \lambda \cup 1^{b-q}$ then there is a unique partition β^λ of $b - q$ such that*

$$\beta^\lambda \trianglerighteq \sigma \qquad \textit{iff} \quad \beta \trianglerighteq \lambda \cup \sigma.$$

Proof. Let $\beta^\lambda = \bigvee\{\sigma \mid \beta \trianglerighteq \sigma \cup \lambda\}$. Since $\beta \trianglerighteq \lambda \cup 1^{b-q}$, $\{\sigma \mid \beta \trianglerighteq \sigma \cup \lambda\}$ is nonempty. It is clear from the definition of β^λ that if $\beta \trianglerighteq \lambda \cup \sigma$ then $\beta^\lambda \trianglerighteq \sigma$. To prove the converse it is enough by Proposition I.13 to show that $\beta \trianglerighteq \lambda \cup \beta^\lambda$. But this is true since if $\beta \trianglerighteq \lambda \cup \sigma$ and $\beta \trianglerighteq \lambda \cup \tau$ then $\beta \trianglerighteq (\lambda \cup \sigma) \vee (\lambda \cup \tau) = \lambda \cup (\sigma \vee \tau)$ by Lemma I.11.

We note that if λ has only one nonzero entry (i.e. $\lambda = q$) then if it exists β^λ is easy to describe. Namely, pick j so tht $\beta(j) \geqslant q > \beta(j+1)$ and then let

$$\beta^\lambda(i) = \begin{cases} \beta(i) & \text{if } i < j, \\ \beta(j) + \beta(j+1) - q & \text{if } i = j, \\ \beta(i+1) & \text{if } i > j. \end{cases}$$

PROPOSITION I.15. *Let α and π split n and β and λ split k. If $\alpha \cup \beta = \pi \cup \lambda$, $\alpha \trianglerighteq \pi$ and $\beta \trianglerighteq \lambda$ then $\alpha = \pi$ and $\beta = \lambda$.*

Proof. If $\alpha \cap \pi$ and $\beta \cap \lambda$ were both empty then $\alpha = \lambda$ and $\beta = \pi$ so $\alpha \trianglerighteq \pi = \beta \trianglerighteq \lambda = \alpha$ and $\alpha = \pi = \beta = \lambda$.

Thus we can assume $\alpha \cap \pi = \sigma$ is not empty. Then $\alpha = \alpha' \cup \sigma$ and $\pi = \pi' \cup \sigma$, and since $\alpha' \cup \sigma \trianglerighteq \pi' \cup \sigma$ by the previous proposition $\alpha' \trianglerighteq \pi'$. But now $\alpha' \cup \beta = \pi' \cup \lambda$, $\alpha' \trianglerighteq \pi'$ and $\beta \trianglerighteq \lambda$ so the proof is completed by induction on the number of nonzero parts in $\alpha \cup \beta$, the case of two parts being trivial.

Finally we state a few results which concern the well-known Frobenius–Schur isomorphism [16, 3] between $R(S_n)$ and the module (over Z) of homogeneous degree n symmetric polynomials. To each irrep $\{\pi\}$ of $S(n)$ is associated a symmetric polynomial e_π of degree n, sometimes also denoted by $\{\pi\}$, called the Schur function associated to π. This determines the isomorphism because $\{e_\pi : \pi \vdash n\}$ is an integral basis for the degree n symmetric polynomials [2, 7]. If $\pi \vdash n$ and $\sigma \vdash k$ then $e_\pi e_\sigma = \sum g(\pi, \sigma, \beta)\, e_\beta$ where the sum is over all $\beta \vdash n + k$. The numbers $g(\pi, \sigma, \beta)$ can be determined combinatorially by the Littlewood–Richardson rule (see [2, p. 94]). In particular the following special case can be easily worked out.

PROPOSITION I.16. *$g(n, k, \beta) = 1$ if $\beta = (n + k - r, r)$ for some r with $0 \leqslant r \leqslant \min(n, k)$, otherwise $g(n, k, \beta) = 0$.*

Proof. See [2, IV, p. 92].

Corresponding to multiplication of symmetric functions there is an outer product mapping $R(S(n)) \times R(S(k))$ into $R(S(n + k))$. Namely, for reps α, γ of $S(n)$ and $S(k)$ the outer product $\alpha \# \gamma$ is defined to be $\mathrm{Ind}_{S(n)\times S(k)}^{S(n+k)} \alpha \otimes \gamma$. This makes $R = \bigoplus_0^\infty R(S_{(n)})$ into a commutative ring and the isomorphism above preserves multiplication according to the following theorem.

THEOREM I.17. $\{\pi\} \# \{\sigma\} = \sum g(\pi, \sigma, \beta)\{\beta\}$ $(\beta \vdash n + k)$.

Proof. See [4, Theorem 3.31, p. 62] or [16, Chap. 3].

PROPOSITION I.18. $\Delta(\pi) \# \Delta(\sigma) = \Delta(\pi \cup \sigma)$.

Proof. Since $S(\pi) \times S(\sigma) = S(\pi \cup \sigma)$, this is clear from Proposition I.4 and transitivity of induction.

COROLLARY I.19. $\Delta(\pi) = \{\pi(1)\} \# \{\pi(2)\} \# \cdots \# \{\pi(p)\}$.

Proof. It is clear from the definitions that $\Delta(\pi(1)) = \{\pi(1)\} = 1$.

The following result is just a restatement of a known fact concerning symmetric polynomials since $\langle \Delta(\pi), \{\alpha\} \rangle = K_\pi^\alpha$ is the classical Kostka coefficient of π and α [7, 16].

PROPOSITION I.20. *Let β be a partition of $n + k$ then $\langle \Delta(\pi \cup k), \{\beta\} \rangle = \sum \langle \Delta(\pi), \{\alpha\} \rangle \, g(\alpha, k, \beta)(\alpha \vdash n)$.*

We note in conclusion that the facts expressed in the last three propositions have much to do with the (K, ρ, λ) structures introduced by Foulkes in [8].

II. CHARACTERS OF THE HYPEROCTAHEDRAL GROUPS

First we construct all the irreps of $B(n)$. Let $E(n)$ be the subgroup of $n \times n$ diagonal matrices with ± 1 diagonal entries. Let $e(i)$ be that member of $E(n)$ with -1 in the ith diagonal position and $+1$ elsewhere. Clearly $E(n)$ is generated by $\{e(i)\}$. For each integer j with $0 \leqslant j \leqslant n$ define a character $\chi(j)$ of $E(n)$ by

$$\chi(j)[e(i)] = \begin{cases} -1 & \text{if } i \leqslant j, \\ +1 & \text{if } i > j. \end{cases}$$

It is easy to show that any irreducible character of $E(n)$ is conjugate by an element of $S(n)$ to precisely one $\chi(j)$.

The group $B(n)$ is the semidirect product of $S(n)$ by $E(n)$. For any group G which is the semidirect product of a subgroup H by an abelian normal subgroup A, as a consequence of the Mackey subgroup theorem, the irreps of G are all uniquely $\mathcal{O}(\chi, \rho)$ where χ is a representative from an orbit of H acting on the one dimensional characters of A, ρ is an irrep of H_χ the stabilizer of χ in H and $\mathcal{O}(\chi, \rho)$ is the result of inducing $\chi \otimes \rho$ from AH_χ up to G (see Serre [5, Proposition 25, p. 78]). Since the stabilizer of $\chi(j)$ in $S(n)$ is $S(j) \times S(n-j)$ any irrep of $B(n)$ is uniquely of the form $\operatorname{Ind}_{E(n)\times[S(j)\times S(n-j)]}^{B(n)} \chi(j) \otimes \{\pi\} \otimes \{\lambda\}$ where $(\pi; \lambda)$ is a double partition for n and π partitions j. We will denote this irreducible representation (and the associated character) by $\{\pi; \lambda\}$. This construction of the irreducibles is essentially that used by Osima [12].

The canonical representation of $B(n)$ operating on $\mathbb{C}^n$ (or $\mathbb{R}^n$) is $\{1; n-1\}$ since $E(n) \times S(1) \times S(n-1)$ acts as $\chi(1) \otimes \{1\} \otimes \{n-1\}$ on the one dimensional subspace spanned by the first standard basis vector. It is easily seen that the one dimensional characters of $B(n)$ are $\{0; n\}$ (trivial), $\{n; 0\}$ ($\chi(n)$ on $E(n)$ and trivial on $S(n)$), $\{0; 1^n\}$ (trivial on $E(n)$ and alternating on $S(n)$), and $\{1^n; 0\}$ (determinant of the canonical representation).

Multiplication by these characters has the following simple effect.

PROPOSITION II.1. (i) $\{\pi; \lambda\} \cdot \{0; n\} = \{\pi; \lambda\}$,

(ii) $\{\pi; \lambda\} \cdot \{n; 0\} = \{\lambda; \pi\}$,

(iii) $\{\pi; \lambda\} \cdot \{0; 1^n\} = \{\pi^*; \lambda^*\}$,

(iv) $\{\pi; \lambda\} \cdot \{1^n; 0\} = \{\lambda^*; \pi^*\}$.

Proof. It is sufficient to check equality (or equivalence) on $E(n) \times S(j) \times S(n-j)$ before inducing and for this note that $S(j) \times S(n-j)$ is conjugate to $S(n-j) \times S(j)$ in $S(n)$ and $\{1^n\} \cdot \{\pi\} = \{\pi^*\}$.

For the remainder of this section $(\pi; \lambda)$ and $(\alpha; \beta)$ will be double partitions for a fixed integer n, with π partitioning p and α partitioning a. If μ is a partition of m let $B(\mu)$ be the subgroup of $B(m)$ which is the semidirect product of $S(\mu)$ by $E(m)$. Let $F(p)$ be the subgroup of $E(n)$ generated by $\{e(i) \mid p < i \leqslant n\}$. There is a natural identification of the direct product $S(\pi) \times B(\lambda)$ with the subgroup of $B(n)$ which is the semidirect product of $S(\pi) \times S(\lambda)$ by $F(p)$. We will usually denote the latter by $S(\pi) \times B(\lambda)$. (The direct product $B(\pi) \times B(\lambda)$ is called a "generalized Young subgroup" by Puttaswamaiah [13]). We denote by $\Delta(\pi; \lambda)$ the permutation character $\operatorname{Ind}_{S(\pi)\times B(\lambda)}^{B(n)} 1$. For the special cases where λ is a partition with only one non-zero entry (i.e. $\lambda = n - p$) the subgroups $S(\pi) \times B(\lambda)$ are the parabolic subgroups of $B(n)$ and hence the parabolic characters of $B(n)$ are just $\{\Delta(\pi; n-p)\}$.

The remainder of this paper studies these permutation characters and their relationship to the irreducible characters.

We begin by defining a linear order, denoted by $\leqslant$, on the set of double partitions of n. We write $(\pi; \lambda) \leqslant (\alpha; \beta)$ iff one of the following holds.

(i) $p < a$,

(ii) $p = a$ and $\pi < \alpha$,

(iii) $\pi = \alpha$ and $\lambda \leqslant \beta$.

This is just one lexicographic extension of the linear order $\leqslant$ defined for ordinary partitions in Section I. The rest of this section is concerned with the following theorem.

THEOREM II.2. *If* $p < a$ *then* $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle = 0$ *and if* $p = a$ *then* $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle = \langle \Delta(\pi), \{\alpha\}\rangle \cdot \langle \Delta(\lambda); \{\beta\}\rangle$.

Before proving this theorem we note the following consequences.

COROLLARY II.3. (i) *If* $(\pi; \lambda) < (\alpha; \beta)$ *then* $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle = 0$,

(ii) $\langle \Delta(\pi; \lambda), \{\pi; \lambda\}\rangle = 1$.

Proof. Apply Theorem I.7.

COROLLARY II.4. *The set* $\{\Delta(\pi; \lambda) : (\pi; \lambda)$ *a double partition for* $n\}$ *of permutation characters is an integral basis for the representation ring of* $B(n)$.

Proof. Linearly order both the set of irreducible characters $\{\pi; \lambda\}$ and the set of permutation characters $\Delta(\pi; \lambda)$ by using $\leqslant$. Let A be the matrix with entries $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle$ with the rows indexed by the $\Delta(\pi; \lambda)$ listed in ascending order and columns indexed by the $\{\pi; \lambda\}$ listed in ascending order. Then Corollary II.3 says that A is an integral triangular matrix with 0's above the diagonal and 1's on it. Thus A has an integral inverse and so the $\{\alpha; \beta\}$ are integral combinations of the $\Delta(\pi; \lambda)$.

To prove the theorem we need to first establish a preliminary lemma. Let U be a set of double coset representatives in $B(n)$ for $S(\pi) \times B(\lambda)$ on the left and $B(a) \times B(n-a)$ on the right. For each $u \in U$ let $Q(u)$ be the representation of $u[B(a) \times B(n-a)]\, u^{-1} \cap [S(\pi) \times B(\lambda)]$ defined by $Q(u)[w] = (\chi(a) \otimes \{\alpha\} \otimes \{\beta\})[u^{-1}wu]$, then using Frobenius reciprocity (Theorem I.1) and the Mackey Subgroup Theorem (Theorem I.2) we have the following.

LEMMA II.5. $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle = \sum \langle 1, Q(u)\rangle \; (u \in U)$.

Proof of Theorem II.2. Since $E(n) \subseteq B(a) \times B(n-a)$ and $B(n) = S(n)\, E(n)$, we can assume that $U \subseteq S(n)$. We can also assume that the identity element is in U.

If $p < a$ then $\langle 1, Q(u)\rangle = 0$ for every u. Namely, there is then some j with $1 \leqslant j \leqslant a$ such that $u[e(j)]\, u^{-1}$ is in $\{e(i) \mid p < i \leqslant n\}$ because conjugation by an element of $S(n)$ permutes the set $\{e(i) \mid 1 \leqslant i \leqslant n\}$, and $|\{u[e(i)]\, u^{-1}\colon 1 \leqslant i \leqslant i \leqslant a\}| = a > p = |\{e(i)\colon 1 \leqslant i \leqslant p\}|$. But then $u[e(j)]\, u^{-1} \in u[B(a) \times$

$B(n-a)]\,u^{-1} \cap [S(\pi) \times B(\lambda)]$ so $Q(u)(u[e(j)]\,u^{-1})$ is defined and $(\chi(a) \otimes \{\alpha\} \otimes \{\beta\})\,[e(j)] = -(\text{identity})$ since $\chi(a)[e(j)]$ acts as multiplication by -1. Thus $Q(u)$ can have no trivial subrepresentations and so $\langle 1, Q(u)\rangle = 0$ for each $u \in U$. Hence $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle = 0$.

Now suppose $p = a$. For any $u \in U$ unless $\{u[e(i)]\,u^{-1}: 1 \leqslant i \leqslant a = p\} = \{e(i) \mid 1 \leqslant i \leqslant a = p\}$ there is a j as above and so $\langle 1, Q(u)\rangle = 0$. But this condition is equivalent to $u \in S(a) \times S(n-a) \subseteq B(a) \times B(n-a)$ and thus by our choice of U this happens only if u is the identity element of $B(n)$. Thus

$$\begin{aligned}\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle &= \langle 1, Q(\mathrm{id})\rangle \\ &= \langle 1, \operatorname{Res}^{B(a)\times B(n-a)}_{S(\pi)\times B(\lambda)} \chi(a) \otimes \{\alpha\} \otimes \{\beta\}\rangle \\ &= \langle 1, \operatorname{Res}^{S(a)\times S(n-a)}_{S(\pi)\times S(\lambda)} \{\alpha\} \otimes \{\beta\}\rangle\end{aligned}$$

since $\chi(a)$ is trivial on $F(a) = F(p)$, that is, the $F(p)$ part of $B(\lambda)$ operates trivially in $\chi(a) \otimes \{\alpha\} \otimes \{\beta\}$. But using Frobenius reciprocity (Theorem I.1), Theorem I.4 and the remark following Theorem I.2, $\langle 1, \operatorname{Res}^{S(a)\times S(n-a)}_{S(\pi)\times S(\lambda)} \{\alpha\} \otimes \{\beta\}\rangle = \langle \Delta(\pi), \{\alpha\}\rangle\langle \Delta(\lambda), \{\beta\}\rangle$ where the inner products on the right side are with respect to $S(a)$ and $S(n-a)$, and we are done.

III. The Irreducible Representations Appearing in $\Delta(\pi; \lambda)$

Let $(\pi; \lambda)$ be a double partition of n with π partitioning p and having q non-zero parts. In this section we will express $\langle \Delta(\pi; \lambda), \{\rho; \tau\}\rangle$ in terms of inner products involving $\{\rho\}$ and $\{\tau\}$.

We first introduce what we will call, as for the symmetric groups, the outer product of two representations [4, p. 52]. If Ω is a representation of $B(k)$ and Λ a representation of $B(n-k)$ then $\operatorname{Ind}^{B(n)}_{B(k)\times B(n-k)} \Omega \otimes \Lambda$ is a representation of $B(n)$, the outer product of Ω and Λ, denoted $\Omega \# \Lambda$. It is easily checked that $\#$ is associative and commutative.

Lemma III.1. $\{\pi; \lambda\} = \{\pi; 0\} \# \{0; \lambda\}$.

Proof.

$$\begin{aligned}\{\pi; 0\} \# \{0; \lambda\} &= \operatorname{Ind}^{B(n)}_{B(p)\times B(n-p)}[\chi(p) \otimes \{\pi\}] \otimes [\chi(0) \otimes \{\lambda\}] \\ &= \operatorname{Ind}^{B(n)}_{E(n)\times S(p)\times S(n-p)} \chi(p) \otimes \{\pi\} \otimes \{\lambda\}.\end{aligned}$$

Recall from Section I, that if α is a partition of a and μ is a partition of $n+a$ then $g(\pi, \alpha, \mu)$ is the number of times the Schur function associated to μ appears in the product of the Schur functions associated to α and π. Let $(\alpha; \beta)$ be a double partition of m with α partitioning a. The following theorem gives a complete description of the outer product $\{\alpha; \beta\} \# \{\pi; \lambda\}$.

THEOREM III.2. $\{\alpha; \beta\} \# \{\pi; \lambda\} = \sum_{(\mu;\epsilon)} g(\alpha, \pi, \mu)\, g(\beta, \lambda, \epsilon)\{\mu; \epsilon\}$.

Proof. By a routine computation

$$\begin{aligned} \{\alpha; 0\} \# \{\pi; 0\} &= \chi(p+a) \otimes [\mathrm{Ind}_{S_{(p)} \times S_{(a)}}^{S_{(p+a)}} \{\alpha\} \otimes \{\pi\}]. \\ &= \chi(p+a) \otimes \left[\sum_{\mu} g(\alpha, \pi, \mu)\{\mu\}\right] \qquad \text{by Thm. I.17} \\ &= \sum_{\mu} g(\alpha, \pi, \mu)\{\mu; 0\}. \end{aligned}$$

Similarly $\{0; \beta\} \# \{0; \lambda\} = \sum_{\epsilon} g(\beta, \lambda, \epsilon)\{0; \epsilon\}$. Then by Lemma III.1,

$$\begin{aligned} \{\alpha; \beta\} \# \{\pi; \lambda\} &= [\{\alpha; 0\} \# \{\pi; 0\}] \# [\{0; \beta\} \# \{0; \lambda\}] \\ &= \sum_{(\mu;\epsilon)} g(\alpha, \pi, \mu)\, g(\beta, \lambda, \epsilon)[\{\mu; 0\} \# \{0; \epsilon\}] \\ &= \sum_{(\mu;\epsilon)} g(\alpha, \pi, \mu)\, g(\beta, \lambda, \epsilon)\{\mu; \epsilon\}. \end{aligned}$$

Another pleasant and nearly obvious property of $\#$ is the following.

LEMMA III.3. *For any integer j,*

(i) $\Delta(\pi; \lambda \cup j) = \Delta(\pi; \lambda) \# \Delta(0; j)$,

(ii) $\Delta(\pi \cup j; \lambda) = \Delta(\pi; \lambda) \# \Delta(j; 0)$,

(iii) $\Delta(\pi; \lambda) = \Delta(\pi; 0) \# \Delta(0; \lambda)$,

(iv) $\Delta(\pi \cup \alpha; \lambda \cup \beta) = \Delta(\pi; \lambda) \# \Delta(\alpha; \beta)$.

This then allows us to show the following.

PROPOSITION III.4. (i) $\{\pi(1); 0\} \# \{\pi(2); 0\} \# \cdots \# \{\pi(q); 0\} = \sum_{\alpha} \langle \Delta(\pi), \{\alpha\}\rangle \{\alpha; 0\}$,

(ii) $\Delta(0; \pi) = \{0; \pi(1)\} \# \{0; \pi(2)\} \# \cdots \# \{0; \pi(q)\} = \sum_{\alpha} \langle \Delta(\pi), \{\alpha\}\rangle \{0; \alpha\}$,

(iii) $\{1^{\pi(1)}; 0\} \# \{1^{\pi(2)}; 0\} \# \cdots \# \{1^{\pi(q)}; 0\} = \sum_{\alpha} \langle \Delta(\pi), \{\alpha\}\rangle \{\alpha^*; 0\}$,

(iv) $\{0; 1^{\pi(1)}\} \# \{0; 1^{\pi(2)}\} \# \cdots \# \{0; 1^{\pi(q)}\} = \sum_{\alpha} \langle \Delta(\pi), \{\alpha\}\rangle \{0; \alpha^*\}$.

Proof. Since $\operatorname{Res}^{B(n)}_{B(k)\times B(n-k)}\{n;0\} = \{k;0\}\otimes\{n-k;0\}$ and similarly for $\{0;1^n\}$ and $\{1^n;0\}$ we can obtain (ii), (iii) and (iv) by multiplying both sides of (i) by the characters $\{n;0\}$, $\{0;1^n\}$ and $\{1^n;0\}$ respectively and using Proposition II.1.

To show (i), we note that Theorem III.2 insures that only irreps of the form $\{\alpha;0\}$ can appear in the left side. To show that $\langle\{\pi(1);0\}\#\{\pi(2);0\}\#\cdots\#\{\pi(q);0\},\{\alpha;0\}\rangle = \langle\Delta(\pi),\{\alpha\}\rangle$ we will use induction on q. If $q=1$ then the equation becomes $\langle\{\pi(1);0\},\{\alpha;0\}\rangle = \langle\{\pi(1)\},\{\alpha\}\rangle$ and both sides are 1 or 0 depending on whether $\pi(1)=\alpha$ or not. Assume the conclusion holds for all partitions with $\leqslant t$ nonzero entries and let π have $t+1$ nonzero entries. Let $\bar{\pi}$ be the partition of $p-\pi(t+1)$ obtained by deleting the entry $\pi(t+1)$ from π, then by the induction hypothesis $\{\pi(1);0\}\#\cdots\#\{\pi(t);0\} = \sum_\beta\langle\Delta(\bar{\pi}),\{\beta\}\rangle\{\beta;0\}$. Thus

$$\begin{aligned}
&\langle\{\pi(1);0\}\#\cdots\#\{\pi(t+1);0\},\{\alpha;0\}\rangle\\
&\quad=\sum_\beta\langle\Delta(\bar{\pi}),\{\beta\}\rangle\langle\{\beta;0\}\#\{\pi(t+1);0\},\{\alpha;0\}\rangle\\
&\quad=\sum_\beta\langle\Delta(\bar{\pi}),\{\beta\}\rangle\, g(\beta,\pi(t+1),\alpha) \qquad \text{by Thm. III.2}\\
&\quad=\langle\Delta(\pi),\{\alpha\}\rangle \qquad \text{by Prop. I.20.}
\end{aligned}$$

We return now to our initial objective. Let $M=(m(1),m(2),\ldots,m(q))$ be a sequence of nonnegative integers. We write $M\subset\pi$ if $m(i)\leqslant\pi(i)$ for all i. Let $(\alpha;\beta)$ be a double partition of n with $\alpha\vdash a$.

THEOREM III.5. $\langle\Delta(\pi;\lambda),\{\alpha;\beta\}\rangle = \sum_{M\subset\pi}\langle\Delta(M),\{\alpha\}\rangle\langle\Delta((\pi-M)\cup\lambda),\{\beta\}\rangle$ *where* $\pi-M$ *is the sequence* $(\pi(1)-m(1),\pi(2)-m(2),\ldots,\pi(q)-m(q))$.

To show this we need the following special cases.

LEMMA III.6. (i) $\Delta(0;n)=\{0;n\}=1$,

(ii) $\Delta(n;0)=\sum\{r;n-r\}=\sum\{r;0\}\#\{0;n-r\}(0\leqslant r\leqslant n)$.

Proof. (i) This is clear from the definitions. (ii) Since U (the set of double coset representatives in Lemma II.5) contains only the identity element in this case, by Lemma II.5 $\langle\Delta(n;0),\{\alpha;\beta\}\rangle = \langle 1,Q(id)\rangle = \langle 1,\{\alpha\}\otimes\{\beta\}\rangle$. But both 1 and $\{\alpha\}\otimes\{\beta\}$ are irreps of $S(a)\times S(n-a)$ and thus the inner product is zero unless $\{\alpha\}\otimes\{\beta\}=1$ (that is, $\alpha=a$ and $\beta=n-a$) in which case the inner product is 1.

Proof of Theorem III.5.

$$\begin{aligned}
\Delta(\pi;\lambda) &= \Delta(\pi(1);0)\ \#\ \Delta(\pi(2);0)\ \#\ \cdots\ \#\ \Delta(\pi(q);0)\ \#\ \Delta(0;\lambda) \quad \text{by Lemma III.3}\\
&= \sum_{M\subset\pi} \{m(1);\pi(1)-m(1)\}\ \#\ \{m(2);\pi(2)-m(2)\}\ \#\ \cdots\\
&\qquad \#\ \{m(q);\pi(q)-m(q)\}\ \#\ \Delta(0;\lambda) \quad \text{by Lemma III.6}\\
&= \sum_{M\subset\pi} [\{m(1);0\}\ \#\ \{m(2);0\}\ \#\ \cdots\ \#\ \{m(q);0\}]\\
&\qquad \#\ [\Delta(0;\pi(1)-m(1))\ \#\ \cdots\ \#\ \Delta(0;\pi(q)-m(q))\ \#\ \Delta(0;\lambda)]\\
&\qquad \text{by Lemmas III.1 and III.6}\\
&= \sum_{M\subset\pi} [\{m(1);0\}\ \#\ \{m(2);0\}\ \#\ \cdots\ \#\ \{m(q);0\}]\ \#\ \Delta(0;(\pi-M)\cup\lambda)\\
&\qquad \text{by Lemma III.3}\\
&= \sum_{M\subset\pi}\Big[\sum \langle\Delta(M),\{\alpha\}\rangle\{\alpha;0\}\Big]\\
&\qquad \#\Big[\sum_{\beta}\langle\Delta((\pi-M)\cup\lambda),\{\beta\}\rangle\{0;\beta\}\Big] \quad \text{by Prop. III.4}\\
&= \sum_{(\alpha;\beta)}\Big[\sum_{M\subset\pi}\langle\Delta(M),\{\alpha\}\rangle\langle\Delta((\pi-M)\cup\lambda),\{\beta\}\rangle\Big]\{\alpha;\beta\}\\
&\qquad \text{by Lemma III.1.}
\end{aligned}$$

COROLLARY III.7. *$\langle\Delta(\pi;\lambda),\{\alpha;\beta\}\rangle \neq 0$ iff there exists a partition $\mu\subset\pi$ such that $\alpha\trianglerighteq\mu$ and $\beta\trianglerighteq(\pi-\mu)\cup\lambda$.*

Proof. If such a partition μ exists then $\langle\Delta(\pi,\lambda),\{\alpha;\beta\}\rangle\neq 0$ by Theorems III.5 and I.8. Conversely, if the inner product is non-zero then by Theorem III.5 there is a sequence $M\subset\pi$ such that $\langle\Delta(M),\{\alpha\}\rangle\neq 0$ and $\langle\Delta((\pi-M)\cup\lambda),\{\beta\}\rangle\neq 0$. Let μ be the descending rearrangement of M. By Theorem I.8 $\alpha\trianglerighteq\mu$. Since $\beta\trianglerighteq(\pi-M)\cup\lambda$ we need only show that $(\pi-M)\cup\lambda\trianglerighteq(\pi-\mu)\cup\lambda$ or equivalently by Proposition I.13 that $\pi-M\trianglerighteq\pi-\mu$. Here $\pi-M$ and $\pi-\mu$ have been put in decreasing order as usual. It will suffice to show that if $m(j)<m(j+1)$ and $M'=(m(1),\dots,m(j-1),m(j+1),m(j),m(j+2),\dots,m(q))$ then $\pi-M\trianglerighteq\pi-M'$. By Proposition I.13 we need only show that

$$(\pi(j)-m(j),\pi(j+1)-m(j+1))\trianglerighteq(\pi(j)-m(j+1),\pi(j+1)-m(j))$$

and this follows since π is descending and $m(j)<m(j+1)$.

We conclude this section with a characterization of the irreducible characters $\{\pi;\lambda\}$ using the permutation characters $\Delta(\pi;\lambda)$ and the determinant character $\{1^n;0\}$.

PROPOSITION III.8. *$\{\pi; \lambda\}$ is the unique irreducible character which appears in both $\Delta(\pi; \lambda)$ and $\{1^n; 0\} \cdot \Delta(\lambda^*; \pi^*)$.*

Proof. Suppose $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \neq 0$ and $\langle \Delta(\lambda^*; \pi^*), \{\beta^*; \alpha^*\}\rangle = \langle \{1^n; 0\} \cdot \Delta(\lambda^*; \pi^*), \{\alpha; \beta\}\rangle \neq 0$, then by Corollary II.3, $(\alpha; \beta) \leqslant (\pi; \lambda)$ and $(\beta^*; \alpha^*) \leqslant (\lambda^*; \pi^*)$. Hence $a \leqslant p$ and $n - a \leqslant n - p$, so $p = a$, but then by Theorem II.2 $\langle \Delta(\pi), \{\alpha\}\rangle\langle \Delta(\lambda), \{\beta\}\rangle \neq 0$ and $\langle \Delta(\lambda^*), \{\beta^*\}\rangle\langle \Delta(\pi^*), \{\alpha\}^*\rangle \neq 0$. Thus by Theorem I.8, $\alpha \trianglerighteq \pi$, $\beta \trianglerighteq \lambda$, $\alpha^* \trianglerighteq \pi^*$ and $\beta^* \trianglerighteq \lambda^*$, and then by Proposition I.10 $\alpha \trianglerighteq \pi \trianglerighteq \alpha$ and $\beta \trianglerighteq \lambda \trianglerighteq \beta$, so $\alpha = \pi$ and $\beta = \lambda$.

Note that when $\pi = 0$ this reduces to the corresponding statement for characters of $S(n)$ given by Snapper [14, p. 532].

IV. A Partial Order on Characters

Throughout this section $(\pi; \lambda)$ and $(\alpha; \beta)$ will be double partitions of n with $\pi \vdash p$ and $\alpha \vdash a$. Define an order relation by setting $(\alpha; \beta) \lesssim (\pi; \lambda)$ if $\Delta(\pi; \lambda) - \Delta(\alpha; \beta)$ is zero or a proper character. This is just the restriction to the permutation characters $\Delta(\pi; \lambda)$ of the partial order on all characters (or all class functions) obtained by taking the cone generated by the irreps as the positive cone. By Theorem I.8 the above definition of $\lesssim$ is one analogue of the dominance order for single partitions. We now consider the analogues of some of the equivalent definitions of dominance.

PROPOSITION IV.1. (i) *If $(\alpha; \beta) \lesssim (\pi; \lambda)$ then $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \neq 0$.*

(ii) *If $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \neq 0$ then $\langle \Delta(\pi; \lambda), \{1^n; 0\} \cdot \Delta(\beta^*, \alpha^*)\rangle \neq 0$.*

Proof. (i) If $(\alpha; \beta) \lesssim (\pi; \lambda)$ then $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \geqslant \langle \Delta(\alpha; \beta), \{\alpha; \beta\}\rangle = 1$ by Correlation II.3. (ii) Since $\langle \{1^n; 0\} \cdot \Delta(\beta^*, \alpha^*), \{\alpha; \beta\}\rangle \neq 0$ by Proposition III.8, $\langle \Delta(\pi; \lambda), \{1^n; 0\} \cdot \Delta(\beta^*; \alpha^*)\rangle \geqslant \langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle > 0$.

Furthermore neither implication reverses since $\alpha = 3$, $\beta = 0$, $\pi = 1$ and $\lambda = 2$ satisfies the conclusion of statement (i) but not the ypothesis, and $\alpha = 1$, $\beta = 2$, $\pi = 0$ and $\lambda = 1^3$ satisfies the conclusion of statement (ii) but not the hypothesis. These two examples also show that neither of the conclusions yields a transitive condition and thus they cannot be used directly to define a partial order. (However, see Mayer [11] for another approach.)

Since $\leqslant$ is a linear order, the following corollary is an immediate consequence of Corollary II.3 and Proposition IV.1.

COROLLARY IV.2. *If $(\alpha; \beta) \lesssim (\pi; \lambda)$ then $(\alpha; \beta) \leqslant (\pi; \lambda)$.*

As might be expected there are close connections between the dominance order $\trianglerighteq$ and the $\lesssim$ order. One of these is the following.

PROPOSITION IV.3. *The following are equivalent.*

(i) $\alpha \trianglerighteq \pi$,

(ii) $(\alpha; 0) \lesssim (\pi; 0)$,

(iii) $(0; \alpha) \lesssim (0; \pi)$.

Proof. We show each of the latter is equivalent to (i).

If $\alpha \trianglerighteq \pi$ then $\Delta(\pi) - \Delta(\alpha)$ is zero or a proper character by Theorem I.8 hence $\Delta(\pi; 0) - \Delta(\alpha; 0) = [\mathrm{Ind}_{S(n)}^{B(n)} \Delta(\pi)] - [\mathrm{Ind}_{S(n)}^{B(n)} \Delta(\alpha)] = \mathrm{Ind}_{S(n)}^{B(n)} [\Delta(\pi) - \Delta(\alpha)]$ is also zero or a proper character. Conversely if $(\alpha; 0) \lesssim (\pi; 0)$ then $\langle \Delta(\pi; 0), \{\alpha; 0\}\rangle \neq 0$ by Proposition IV.1. But $\langle \Delta(\pi), \{\alpha\}\rangle = \langle \Delta(\pi; 0), \{\alpha; 0\}\rangle$ by Theorem II.5 and hence by Theorem I.8, $\alpha \trianglerighteq \pi$. Thus (i) is equivalent to (ii).

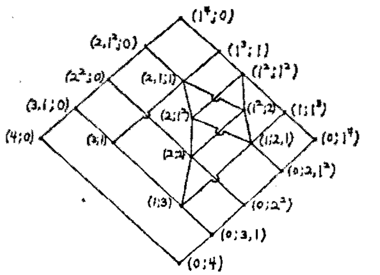
Again assume $\alpha \trianglerighteq \pi$. We wish to show that $\langle \Delta(0; \pi), \{\beta; \lambda\}\rangle \geqslant \langle \Delta(0; \alpha), \{\beta; \lambda\}\rangle$ for all double partitions $(\beta; \lambda)$. If $\beta \neq 0$ then both sides are zero by Corollary II.3. If $\beta = 0$ then by Proposition III.4 and since $\Delta(\pi) - \Delta(\alpha)$ is zero or a proper character, $\langle \Delta(0, \pi), \{0; \lambda\}\rangle = \langle \Delta(\pi), \{\lambda\}\rangle \geqslant \langle \Delta(\alpha), \{\lambda\}\rangle = \langle \Delta(0; \alpha), \{0, \lambda\}\rangle$. Hence (i) implies (iii). Conversely if $(0; \alpha) \lesssim (0; \pi)$ then by Proposition IV.1 $\langle \Delta(0; \pi), \{0; \alpha\}\rangle \neq 0$ hence using Proposition III.4 and Theorem I.8 $\alpha \trianglerighteq \pi$.

The following proposition relates double partitions of n to those of $n + j$ for any positive integer j.

PROPOSITION IV.4. *If* μ, ϵ *are any partitions then* $(\alpha; \beta) \lesssim (\pi; \lambda)$ *implies* $(\alpha \cup \mu; \beta \cup \epsilon) < (\pi \cup \mu; \lambda \cup \epsilon)$.

Proof. By Lemma III.3, $\Delta(\alpha \cup \mu; \beta \cup \epsilon) = \Delta(\alpha; \beta) \# \Delta(\mu; \epsilon)$ and $\Delta(\pi \cup \mu; \lambda \cup \epsilon) = \Delta(\pi; \lambda) \# \Delta(\mu; \epsilon)$, hence if $\Delta(\pi; \lambda) - \Delta(\alpha; \beta)$ is zero or a proper character then so is $\Delta(\pi \cup \mu; \tau \cup \epsilon) - \Delta(\alpha \cup \mu; \beta \cup \epsilon) = [\Delta(\pi; \lambda) - \Delta(\alpha; \beta)] \# \Delta(\mu; \epsilon)$.

Although dominance is always a lattice order, $\lesssim$ is not as the Hasse diagram for $n = 4$ shows ((2; 1^2) and (1^2; 2) have no greatest lower bound or least upper bound).



In this Hasse diagram and in others that have been computed, we see that almost all of the edges (covering pairs) occur in one of two types of intervals, those of the form $[(k; n-k), (1^k; 1^{n-k})]$ and those of the form $[(0; \sigma), (\sigma; 0)]$ where σ is a partition of n. We will show that each of these two classes of intervals forms a striation of the partial order $\lesssim$, that is, a partition into more or less parallel intervals with an order among the intervals. The following theorem characterizes intervals of the first type in terms of the lattice $L^*(j)$ of Section I.

THEOREM IV.5. *The interval* $[(k; n-k), (1^k; 1^{n-k})]$ *is isomorphic to the direct product of* $L^*(k)$ *and* $L^*(n-k)$, *and so it is a lattice.*

Proof. By Corollary IV.2, if $(k; n-k) \lesssim (\pi; \lambda) \lesssim (1^k; 1^{n-k})$ then $(k; n-k) \leqslant (\pi; \lambda) \leqslant (1^k; 1^{n-k})$ so π partitions k and λ partitions $n-k$, and hence any element in the interval is contained in the direct product. Assume α and π both partition k. If we can show that $(\alpha; \beta) \lesssim (\pi; \lambda)$ iff $\alpha \rhd \pi$ and $\beta \rhd \lambda$ then we will be done. Suppose $(\alpha; \beta) \lesssim (\pi; \lambda)$, then $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \neq 0$ by Proposition IV.1 and hence by Theorem II.2 and Theorem I.8 $\alpha \rhd \pi$ and $\beta \rhd \lambda$. Conversely, suppose $\alpha \rhd \pi$ and $\beta \rhd \lambda$, then by Proposition IV.3 $(\alpha; 0) \lesssim (\pi; 0)$ and $(0; \beta) \lesssim (0; \lambda)$. Hence by Proposition IV.4, $(\alpha; \beta) \lesssim (\alpha; \lambda) \lesssim (\pi; \lambda)$.

Within each such interval an exact analogue of condition (ii) of Theorem I.8 holds.

COROLLARY IV.6. *If* α *and* π *partition the same integer then* $(\alpha; \beta) \lesssim (\pi; \lambda)$ *iff* $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \neq 0$.

Proof. Since $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle = \langle \Delta(\pi), \{\alpha\}\rangle\langle \Delta(\lambda), \{\beta\}\rangle$ by Theorem II.2, the result is obvious from Theorem I.8 and the preceding proposition.

The following result is very similar to Corollary III.7 and contains as a special case the preceding Corollary IV.6.

PROPOSITION IV.7. $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \neq 0$ *iff* $\beta \rhd \lambda \cup 1^{p-a}$ *and* $\alpha + \beta^\lambda \rhd \pi$.

Proof. Suppose the inner product is nonzero. By Corollary III.7 there is a μ such that $\mu \subset \pi$, $\alpha \rhd \mu$ and $\beta \rhd (\pi - \mu) \cup \lambda$. Then by Corollary I.14 $\beta^\lambda \rhd \pi - \mu$ and so $\alpha + \beta^\lambda \rhd \mu + (\pi - \mu) \rhd \pi$.

Conversely, if $\beta \rhd \lambda \cup 1^{p-a}$ and $\alpha + \beta^\lambda \rhd \pi$ then by Theorem IV.5 $(\alpha + \beta^\lambda; \lambda) \lesssim (\pi; \lambda)$. Thus to show that $\langle \Delta(\pi; \lambda), \{\alpha; \beta\}\rangle \neq 0$ it is enough to show that $\langle \Delta(\alpha + \beta^\lambda; \lambda), \{\alpha; \beta\}\rangle \neq 0$. But this latter follows directly from Theorem III.5 since in the sum will occur a term with $M = \alpha$ and $\langle \Delta(\alpha), \{\alpha\}\rangle \cdot \langle \Delta(\beta^\lambda \cup \lambda), \{\beta\}\rangle \neq 0$.

The parabolic characters of $B(n)$ are precisely the $\Delta(\pi; n-p)$, so the following is a generalization of a result by Curtis and Benson [18].

COROLLARY IV.8. *If for every parabolic character* $\Delta(\pi; n-p)$, $\langle \Delta(\pi; n-p), \{\alpha; \beta\}\rangle \neq 0$ *iff* $\langle \Delta(\pi; n-p), \{\rho; \tau\}\rangle \neq 0$, *then* $\{\alpha; \beta\} = \{\rho; \tau\}$.

Proof. By the preceding proposition with $\pi = 1^p$ we may conclude that for each choice of p, $\beta \rhd (n-p) \cup 1^{p-a}$ iff $\tau \rhd (n-p) \cup 1^{p-r}$. That is, $\beta(1) = \tau(1)$. Moreover, for each $p \geqslant n - \beta(1)$ and $\pi \vdash p$, $\alpha + \beta^{n-p} \rhd \pi$ iff $\rho + \tau^{n-p} \rhd \pi$. Thus $\alpha + \beta^{n-p} = \rho + \tau^{n-p}$, and if $p = n$ we get $\beta^{n-p} = \beta$ and $\alpha + \beta = \rho + \tau$. In particular $\alpha(1) = \rho(1)$. Assume now that $\beta(i) = \tau(i)$ and $\alpha(i) = \rho(i)$ for all $i \leqslant j$. By remark following Corollary I.14, $\beta^{\beta(j)} = (\beta(1),..., \beta(j-1), \beta(j+1),...)$ and similarly for $\tau^{\beta(j)} = \tau^{\tau(j)}$. Then the jth entry in $\alpha + \beta^{\beta(j)} = \rho + \tau^{\beta(j)}$ gives $\alpha(j) + \beta(j+1) = \rho(j) + \tau(j+1)$, so $\beta(j+1) = \tau(j+1)$ and hence also $\alpha(j+1) = \rho(j+1)$. It follows that $\beta = \tau$ and $\alpha = \rho$.

Since the parabolic characters of $S(n)$ are precisely the $\{\Delta(\pi)\}$, an integral basis, the conclusion of Corollary IV.8 obviously holds for characters of $S(n)$ also. This is not however true for the Weyl groups of type $D(n)$. (The parabolic characters are too symmetric in the irreducible characters.)

We now turn our attention to intervals of the second type. The following proposition characterizes the double partitions in the interval $[(0; \sigma), (\sigma; 0)]$ where σ is a partition of n.

PROPOSITION IV.9. (i) *If* $(\alpha; \beta) \lesssim (\pi; \lambda)$ *then* $(0; \alpha \cup \beta) \lesssim (0; \pi \cup \lambda)$ *and* $(\alpha \cup \beta; 0) \lesssim (\pi \cup \lambda; 0)$.

(ii) *Let* σ *be a partition of* n, *then* $(0; \sigma) \lesssim (\pi; \lambda) \lesssim (\sigma; 0)$ *iff* $\pi \cup \lambda = \sigma$.

Proof. (i) By Theorem III.5 for any partition τ of n, $\langle \Delta(\pi; \lambda), \{0, \tau\}\rangle = \langle \Delta(\pi \cup \lambda), \{\tau\}\rangle$. Thus if $(\alpha; \beta) \lesssim (\pi; \lambda)$ then $\langle \Delta(\pi \cup \lambda), \{\tau\}\rangle \geqslant \langle \Delta(\alpha \cup \beta), \{\tau\}\rangle$ for every τ so by Theorem I.8, $\alpha \cup \beta \rhd \pi \cup \lambda$. Hence by Proposition IV.3 $(0; \alpha \cup \beta) \lesssim (0; \pi \cup \lambda)$ and $(\alpha \cup \beta; 0) \lesssim (\pi \cup \lambda; 0)$.

(ii) Assume $(0; \sigma) \lesssim (\pi; \lambda) \lesssim (\sigma; 0)$ then by the first part of the proposition $(0; \sigma) \lesssim (0; \pi \cup \lambda) \lesssim (0; \sigma)$ and hence $\sigma = \pi \cup \lambda$. Conversely assume $\sigma = \pi \cup \lambda$. By Corollary II.3 $\langle \Delta(0; \sigma), \{\rho; \tau\}\rangle = 0$ for any double partition $(\rho; \tau)$ of n with $\rho \neq 0$ and hence $\langle \Delta(\pi; \lambda), \{\rho; \tau\}\rangle \geqslant \langle \Delta(0; \sigma), \{\rho; \tau\}\rangle$ if $\rho \neq 0$. But as in the proof of the first part $\langle \Delta(\pi; \lambda), \{0, \tau\}\rangle = \langle \Delta(0; \sigma), \{0; \tau\}\rangle$ for every τ, hence $(0; \pi \cup \lambda) = (0; \sigma) \lesssim (\pi; \lambda)$. In particular $(0; \lambda) \lesssim (\lambda; 0)$ and so by Proposition IV.4 $(\pi; \lambda) \lesssim (\pi \cup \lambda; 0) = (\sigma; 0)$. Thus we have proven part (ii).

Thus the type 2 intervals do form a striation. Moreover the mapping sending each $(\pi; \lambda)$ to the top $(\pi \cup \lambda; 0)$ [or to the bottom $(0; \pi \cup \lambda)$] of its type 2 interval is order preserving.

We can use Proposition IV.9 to obtain a nice characterization for part of the order relation $\lesssim$.

PROPOSITION IV.10. *Suppose* $\alpha \subset \pi$. *Then* $(\alpha; \beta) \lesssim (\pi; \lambda)$ *iff* $\alpha \cup \beta \rhd \pi \cup \lambda$.

Proof. If $(\alpha; \beta) \lesssim (\pi; \lambda)$ then by Proposition IV.9 $(0; \alpha \cup \beta) \lesssim (0; \pi \cup \lambda)$ and hence $\alpha \cup \beta \trianglerighteq \pi \cup \lambda$ by Proposition IV.3.

Assume now that $\alpha \cup \beta \trianglerighteq \pi \cup \lambda$. If $M \subset \alpha$ then using Proposition I.12 and induction on the integer partitioned by M, it is easy to show that $(\alpha - M) \cup \beta \trianglerighteq (\pi - M) \cup \lambda$. Hence by Proposition I.8, for any τ and any $M \subset \alpha$, $\langle \Delta((\pi - M) \cup \lambda), \{\tau\}\rangle \geqslant \langle \Delta((\alpha - M) \cup \beta), \{\tau\}\rangle$.

But then for any double partition $(\rho; \tau)$ we have, using Theorem III.5,

$$\begin{aligned}
\langle \Delta(\pi; \lambda), \{\rho; \tau\}\rangle &= \sum_{M \subset \pi} \langle \Delta(M), \{\rho\}\rangle\langle \Delta((\pi - M) \cup \lambda), \{\tau\}\rangle \\
&\geqslant \sum_{M \subset \alpha} \langle \Delta(M), \{\rho\}\rangle\langle \Delta((\pi - M) \cup \lambda), \{\tau\}\rangle \\
&\geqslant \sum_{M \subset \alpha} \langle \Delta(M), \{\rho\}\rangle\langle \Delta((\alpha - M) \cup \beta), \{\tau\}\rangle \\
&= \langle \Delta(\alpha; \beta), \{\rho; \tau\}\rangle.
\end{aligned}$$

Thus $\Delta(\pi; \lambda) - \Delta(\alpha; \beta)$ is zero or a proper character.

The structure of the type 2 intervals can be described combinatorially as follows.

THEOREM IV.11. *The interval* $[(0; \sigma), (\sigma; 0)]$ *consisting of all* $(\pi; \lambda)$ *such that* $\pi \cup \lambda = \sigma$ *is a distributive lattice with the infimum (meet) of* $(\pi; \lambda)$ *and* $(\alpha; \beta)$ *given by* $((\min[\pi(i), \alpha(i)]); (\max[\lambda(i), \beta(i)]))$.

Proof. It is assumed here as always, that the parts of each partition are in descending order. If we define $(\pi; \lambda) \wedge (\alpha; \beta) = (\rho; \tau)$ where $\rho(i) = \min(\alpha(i), \pi(i))$ and $\tau(i) = \max(\beta(i), \lambda(i))$ then it is easy to check that if $\pi \cup \lambda = \alpha \cup \beta = \sigma$ then $\rho \cup \tau = \sigma$. Moreover with $\wedge$ defined this way, the inteval is a sublattice of a product of chains and hence is distributive.

It remains to be shown that the order defined by $\wedge$ is the same as that defined by $\lesssim$. By Proposition IV.10 if $\alpha \cup \beta = \pi \cup \lambda$ and $\alpha(i) \leqslant \pi(i)$ for all i then $(\alpha; \beta) \lesssim (\pi; \lambda)$. Also by using Theorem III.5 we can show that if $\alpha \cup \beta = \pi \cup \lambda$ and if for some j, $\alpha(j) > \pi(j)$, there is a double partition $(\pi; \epsilon)$ such that $\langle \Delta(\alpha; \beta), \{\mu; \epsilon\}\rangle \neq 0$ and $\langle \Delta(\pi; \lambda), \{\pi; \epsilon\}\rangle = 0$. The detailed proof of this involves long computations to determine whether or not one partition dominates another, and is omitted. It is clear that these facts show that $(\pi; \lambda) \wedge (\alpha; \beta)$ is the infimum of $(\pi; \lambda)$ and $(\alpha; \beta)$ in the interval $[(0; \sigma), (\sigma;\ )]$.

Let $Y(\sigma)$ be those nonempty partitions α for which there is a partition α' (possibly 0) such that $\alpha \cup \alpha' = \sigma$. It is easy to show that $Y(\sigma)$ forms a sublattice of the Young lattice (see Berge [19] for definition). Let $\overline{Y}(\sigma)$ be $Y(\sigma)$ with another minimal element 0 adjoined, then Theorem IV.11 shows that the interval $[(0; \sigma), (\sigma; 0)]$ is lattice isomorphic to $\overline{Y}(\sigma)$.

The following proposition shows that the two types of intervals intersect transversally.

PROPOSITION IV.12. *If* $\alpha \cup \beta = \sigma = \pi \cup \lambda$, $\alpha \vdash k$, $\pi \vdash k$ *and* $\alpha \neq \pi$ *then* $(\alpha; \beta)$ *and* $(\pi; \lambda)$ *are unrelated by* $\lesssim$.

Proof. If $(\alpha; \beta) \lesssim (\pi; \lambda)$ then by Theorem IV.5 $\alpha \trianglerighteq \pi$ and $\beta \trianglerighteq \lambda$, and since $\alpha \cup \beta = \pi \cup \lambda$, we have $\alpha = \pi$ and $\beta = \lambda$ by Proposition I.15.

Besides those covering pairs which appear in the two types of intervals discussed above, we know of one other much less common type. An example appears in the Hasse diagram above for $n = 4$ where $(2; 2)$ covers $(1; 3)$. We state without proof the relevant proposition.

PROPOSITION IV.13. *If* $k < r$ *then* $(\alpha \cup k; \beta \cup r) \lesssim (\alpha \cup (k+1); \beta \cup (r-1))$ *for all partitions* α *and* β. *Moreover,* $(\alpha \cup (k+1); \beta \cup (r-1))$ *covers* $(\alpha \cup k; \beta \cup r)$ *iff the following conditions are satisfied*:

(i) α *contains no entry equal to* r,

(ii) β *contains no entry equal to* k,

(iii) *if* $k < j < r$ *then* $\alpha \cup \beta$ *contains no entry equal to* j.

Thus we know of three possibilities if $(\pi; \lambda)$ covers $(\alpha; \beta)$, and there is some evidence that there are no others.

Conjecture 1. There are no other covers.

In Theorem I.8 there is a simple combinatorial description of the dominance order; one which does not require computation with permutation or irreducible characters. From this description Brylawski [6] obtained a simple characterization of the covering relationship in the dominance lattice. We believe, but cannot yet prove, that the following sufficient condition for $(\alpha; \beta) \lesssim (\pi; \lambda)$ is also necessary.

PROPOSITION IV.14. *If there is a partition* ρ *such that* $\alpha \subset \rho$, $\rho \trianglerighteq \pi$, *and* $\alpha \cup \beta \trianglerighteq \rho \cup \lambda$ *then* $(\alpha; \beta) \lesssim (\rho; \lambda) \lesssim (\pi; \lambda)$.

The proof uses Theorem IV.5 and IV.10, and is omitted.

If $(\alpha; \beta)$ and $(\pi; \lambda)$ are both in the same type 1 (or type 2) interval, then by Theorem IV.5 (or Theorem IV.11) the sufficient condition above is also necessary.

Conjecture 2. For double partitions $(\alpha; \beta)$ and $(\pi; \lambda)$, $(\alpha; \beta \lesssim (\pi; \lambda)$ iff there is a partition ρ such that $\alpha \subset \rho$, $\rho \trianglerighteq \pi$, and $\alpha \cup \beta \trianglerighteq \rho \cup \lambda$.

The following proposition, whose proof we omit, shows that Conjecture 2 implies Conjecture 1.

PROPOSITION IV.15. *If* $(\pi; \lambda)$ *covers* $(\alpha; \beta)$ *and* $\alpha \subset \pi$ *then either*:

(i) $\alpha \vdash a$ *and* $\pi \vdash a$, *or*

(ii) $\alpha \cup \beta = \pi \cup \lambda$, *or*

(iii) $(\alpha;\beta)$ *and* $(\pi;\lambda)$ *are a covering pair of the type described in Proposition* IV.13.

We end our discussion of $\lesssim$ with another conjecture prompted by Proposition I.13 and much computational evidence.

Conjucture 3. For all partitions μ, ϵ, the relation $(\alpha;\beta) \lesssim (\pi;\lambda)$ holds iff $(\alpha \cup \mu; \beta \cup \epsilon) \lesssim (\pi \cup \mu; \lambda \cup \epsilon)$.

By Proposition IV.4 the first relation implies the second. We also know the converse if $(\alpha;\beta)$ and $(\pi;\lambda)$ both lie in the same type 1 interval (by Theorem IV.5 and Proposition I.13) or in the same type 2 interval.

References

1. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.
2. D. E. Littlewood, "The Theory of Group Characters and Matrix Representations of Groups," 2nd ed., Oxford Univ. Press (Clarendon), London, 1950.
3. F. D. Murnaghan, "The Theory of Group Representations," The Johns Hopkins Press, Baltimore, 1938.
4. G. de B. Robinson, "Representation Theory of the Symmetric Group," University of Toronto Press, Toronto, 1961.
5. J. Serre, "Représentations Linéaires des Groupes Finis," 2nd ed., Hermann, Paris, 1971.
6. T. Brylawski, The lattice of integer partitions, *Discrete Math.* **6** (1973), 201–219.
7. H. O. Foulkes, A survey of some combinatorial aspects of symmetric functions, *in* "Colloque sur Permutations (July, 1972)," pp. 79–92, Gauthier–Villars, Paris, 1974.
8. H. O. Foulkes, Paths in ordered structures of partitions, *Discrete Math.* **9** (1974), 365–374.
9. P. Hall, The algebra of partitions, *in* "Proceedings of the 4th Canadian Mathematical Congress Banff, 1957," pp. 147–159, 1959.
10. R. A. Liebler and M. R. Vitale, Ordering the partition characters of the symmetric group, *J. Algebra* **25** (1973), 487–489.
11. S. J. Mayer, On the characters of the Weyl group of type C, *J. Algebra* **33** (1975), 59–67.
12. M. Osima, On the representations of the generalized symmetric group, *Math. J. Okayama Univ.* **4** (1954), 39–56.
13. B. M. Puttaswamaiah, Unitary representations of generalized symmetric groups, *Canad. J. Math.* **21** (1969), 28–38.
14. E. Snapper, Group characters and non-negative integral matrices, *J. Algebra* **19** (1971), 520–535.
15. A. Young, Quantitative substitutional analysis V, *Proc. London Math. Soc.* **31** (1930), 273–288.
16. D. Knutson, "λ-Rings and the Representation Theory of the Symmetric Group," Springer Lecture Notes No. 380, 1973.
17. J. S. Frame, Orthogonal group matrices of hyperoctahedral groups, *Nagoya Math. J.* **27** (1966), 585–590.
18. C. T. Benson and C. W. Curtis, On the degrees and rationality of certain characters of finite chevalley groups, *Trans. Amer. Math. Soc.* **165** (1972), 251–273.
19. C. Berge, "Principles of Combinatorics," Academic Press, New York, 1971.